Simple model of a random walk with arbitrarily long memory

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We present a generalization of the persistent random-walk model in which the step at time n depends on the state of the step at time $n-T$, for arbitrary T. This gives rise to arbitrarily long memory effects, yet by an appropriate transformation the model is tractable by essentially the same techniques applicable to the usual persistent random-walk problem. We apply our results to the specific case of delayed ''step'' persistence, and analyze its asymptotic statistical properties.

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Non-Markovian random processes [1,2] have proven to be very valuable models in a wide variety of applications [3]. Yet, in most instances, these models must be tackled numerically. Analytical treatments are rare and usually involve sophisticated mathematics. Here we present, however, an example of a process with arbitrarily long memory that can be handled with a relatively simple formalism and whose characteristic function can be written in closed form. The process under consideration is a generalization of the persistent random walk (PRW). In the normal PRW model, the probability distribution for the direction of the nth step depends on the direction of the $(n-1)$th step [4]. We generalize the model by assigning a ''state'' variable to each step, and assuming that the probability distribution for the nth step depends on the state of the $(n-T)$th step, for arbitrary T. We will refer to this process as the ''random walk with delayed state persistence'' (RWDSP). We apply our results to a particular realization of this model that has been studied in the context of delayed systems as an example of a system exhibiting stochastic resonance [5]. Other systems with delay are of current interest in a variety of fields, e.g., bistable systems [6], coupled oscillators [7], traffic models [8], neural networks [9], and so on. The common feature in these examples is the enrichment of their otherwise simple behaviors by the inclusion (usually justified by the finite speed of signal transmission between the elements of the system) of time delay [10].

The general statement of the RWDSP is as follows. We define the state $S(n)$ of the walker at the nth step as a label function that can take the values ± 1. We denote as $\phi(l)^+$ and $\phi(l)^-$ the step distributions corresponding to each state of the walker. Then, given a set of T initial steps (with their corresponding state labels), we allow the state of the walker to evolve according to the following transition probabilities:

$$p(S_\tau|S_{\tau-T})=\tfrac{1}{2}(1+S_\tau\varepsilon),\quad S_{\tau-T}=+1;$$
$$p(S_\tau|S_{\tau-T})=\tfrac{1}{2}(1-S_\tau\varepsilon'),\quad S_{\tau-T}=-1. \tag{1}$$

Once the state of the walker at the nth time step is determined, the nth step is chosen from the corresponding step distribution.

The transition probabilities given in Eq. (1), which are the most general for this problem, are completely defined by the ''persistence parameters'' ε, ε' and the ''delay parameter'' T. Clearly, if $\varepsilon=\varepsilon'=0$, all memory effects are lost and we recover a Markovian random walk with step distribution given by $\Phi(l)=\frac{1}{2}[\phi^+(l)+\phi^-(l)]$.

It should be noticed that, given the way the process has been defined, the persistence occurs within the states of the random walker and it will be reflected in the actual motion of the walker through the choice of step distributions corresponding to each state. If, for example, the step distributions are equal, i.e., $\phi^+(l)=\phi^-(l)$, then the persistence effects within the states of the walker are irrelevant for its motion, for any values of the persistence and delay parameters, and the motion reduces, once again, to a Markovian random walk.

The model also reduces to the usual (symmetric) PRW if we choose $\varepsilon=\varepsilon'\neq 0$, the step distributions as $\phi^{\pm}(l)=\delta(l\mp 1)$, and the delay as $T=1$. This choice of step distributions allows the identification of the state with the actual step, thus the persistence among states carries over directly to persistence among steps. Under the same choice of step distributions, we can choose values of $T>1$, to yield generalized, arbitrarily delayed, random walks with persistence among steps. We will study this case in more detail below.

If either $\varepsilon=1$ or $\varepsilon'=1$, but not both, then the pure $S=1$ or $S=-1$ state becomes absorbing, and the process ends up in the corresponding state after a finite number of steps with probability 1. Thereafter, the process becomes equivalent to a Markovian random walk with step distribution corresponding to that of the absorbing state.

The case of extreme persistence is achieved when $\varepsilon=\varepsilon'=1$. In this case we obtain perfectly periodic behavior in the states of the walker, with period T.

The usual approach to deal with finite-memory random walks is to pose them as a Markovian ''multistate random walk'' [4]. For the case at hand, this would require defining as ''states'' every possible combination of $T+1$ labels, and computing the appropriate transfer matrices among them. The analytical solution of the system involves manipulations of $2^{T+1}\times 2^{T+1}$ matrices, a formidable task if we wish to take T arbitrarily large. Thus, we take a different approach.

The quantity of interest is the probability distribution for the position of the walker after τ steps. To compute this quantity, we make the construction shown in Fig. 1. In the figure we have placed the state labels of the initial T steps in

$N=[\tau/T]$	S(NT+1)	.....	S(NT+q)		
	⋮	⋮	⋮	⋮	⋮
$N=2$	S(2T+1)	S(2T+2)	S(2T+3)	.....	S(3T)
$N=1$	S(T+1)	S(T+2)	S(T+3)	.....	S(2T)
$N=0$	S(1)	S(2)	S(3)	.....	S(T)

FIG. 1. Construction for evaluating the position of the RWDSP as the sum of T independent generalized random walks, q of which have given $N+1$ steps, $T-q$ have given N steps. Note that the state of each step depends precisely on the state that is below it in this construction.

the bottom row, the next T steps on the second row, and so on. N is the integer part of τ/T and q is defined through the relation $\tau = NT+q$. The position of the walker at time τ is equal to the sum of all the individual steps taken between time 0 and time τ. Obviously, this sum can be carried out in any order; in particular, we choose to add column by column. The idea is that, by construction, each step is over the step that preceded it T time steps before, which is precisely the step on which the transition matrix depends.

Thus, viewed as stochastic processes, the sums over each column are independent, unit delay, generalized persistent random walks. The total sum is therefore the sum of T independent generalized persistent random walks, q of which have given $N+1$ steps, and $T-q$ have given N steps (these processes are generalizations of the usual PRW in the sense that the persistence is, once again, within the states of the walker, and these states determine the step distribution). The distribution of the sum is simply the convolution of the probability distributions of these generalized persistent random walks. It is also clear that since this problem has memory, to determine the solution, T ''initial'' conditions must be specified, namely the first row of steps (actually, one must also specify the initial position of the walker, but we will always assume this to be zero). Each of these initial steps serves as the initial condition for the process that lies above them in the rearrangement we have made. Strictly speaking, the resulting probability distribution will, therefore, be a function of all the initial conditions, but in the following we will omit the specific dependence in order to avoid a cumbersome notation. Thus, we write $P_\tau(x)$ to denote the probability of finding the walker at x after τ steps, for a given configuration of the initial T states. In view of the discussion above, if the initial states are independent identically distributed variables, the characteristic function associated with $P_\tau(x)$ will be given by

$$\hat{P}_\tau(\theta) = \hat{p}_{NT+q}(\theta) = [\hat{p}_{N+1}(\theta)]^q [\hat{p}_N(\theta)]^{T-q}, \tag{2}$$

where $\hat{p}_N(\theta)$ is the characteristic function of the N-step, unit delay, generalized persistent random walk, which can be evaluated by a simple adaptation of the technique to deal with the usual PRW [4]: We define $a_N(x)$ as the probability of finding this walker at position x at step N, having reached the site from a step taken in the state $S(N-1)=1$, and $b_N(x)$ as the probability of finding the walker at position x at step N, having reached the site from a step taken in the state $S(N-1)=-1$. The characteristic functions for these quantities satisfy the following coupled recursion relations

$$\hat{a}_N(\theta) = \gamma \hat{a}_{N-1}(\theta)\hat{\phi}^+(\theta) + \mu \hat{b}_{N-1}(\theta)\hat{\phi}^+(\theta), \tag{3}$$

$$\hat{b}_N(\theta) = \gamma' \hat{b}_{N-1}(\theta)\hat{\phi}^-(\theta) + \mu' \hat{a}_{N-1}(\theta)\hat{\phi}^-(\theta), \tag{4}$$

where the constants γ, γ', μ, and μ' are given in terms of the persistence parameters through

$$\gamma \equiv \tfrac{1}{2}(1+\varepsilon), \quad \mu \equiv \tfrac{1}{2}(1-\varepsilon'),$$
$$\gamma' \equiv \tfrac{1}{2}(1+\varepsilon'), \quad \mu' \equiv \tfrac{1}{2}(1-\varepsilon). \tag{5}$$

Solving the above recursion relations, the general form of $\hat{p}_N(\theta) = \hat{a}_N + \hat{b}_N$ may be written as

$$\hat{p}_N(\theta) = A\hat{f}_+^N(\theta) + B\hat{f}_-^N(\theta), \tag{6}$$

where

$$\hat{f}_\pm = \tfrac{1}{4}\{[\hat{\phi}^+(\theta) + \hat{\phi}^-(\theta)] + [\varepsilon\hat{\phi}^+(\theta) + \varepsilon'\hat{\phi}^-(\theta)]\} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4}\{[\hat{\phi}^+(\theta) + \hat{\phi}^-(\theta)] + [\varepsilon\hat{\phi}^+(\theta) + \varepsilon'\hat{\phi}^-(\theta)]\}^2 - 2\hat{\phi}^+(\theta)\hat{\phi}^-(\theta)(\varepsilon+\varepsilon')}. \tag{7}$$

The parameters A and B must be determined from initial conditions. To determine the initial conditions, we recall that these generalized, unit delay, persistent random walks were introduced to evaluate sums of steps, which when added give the total displacement of the RWDSP, thus their initial value must be zero. The second condition is given by the choice of states of the first T steps of the RWDSP: denoting by $\pi(s)$ the probability distribution for the state labels of the initial T steps, then the required condition will be given by $p_1(x) = \pi(1)\phi^+(x) + \pi(-1)\phi^-(x)$. With these initial conditions, the parameters A and B are given by

$$A = \frac{\hat{f}_-(\theta) - \hat{p}_1(\theta)}{\hat{f}_-(\theta) - \hat{f}_+(\theta)}, \quad B = \frac{\hat{p}_1(\theta) - \hat{f}_+(\theta)}{\hat{f}_-(\theta) - \hat{f}_+(\theta)}. \tag{8}$$

Relations (6), (7), and (8) together with formula (2) give us the exact characteristic function associated with $P_\tau(x)$.

As an example of the application of the above formalism, in what follows we determine the transport properties of a random walk with delayed *step* persistence. In the normal persistent random-walk model, the probability distribution for the direction of the nth step depends on the direction of the $(n-1)$th step (all steps are of unit length). We apply the above formalism to analyze the case in which the probability distribution for the direction of the nth step depends on the direction of the $(n-T)$th step, for arbitrary T. Essentially the same model was introduced in the context of delayed systems as an example of a system exhibiting stochastic resonance [5]. Our principal interest will be the determination of the asymptotic transport properties of the process.

As mentioned earlier, to describe this process we choose the step distributions as $\phi^{\pm}=\delta(x\mp 1)$. For the distribution of the initial T steps, we will focus on the random symmetric case, which corresponds to the situation in which the walker initially performs a T-step unbiased ''drunkard's'' walk; other cases can be treated in exactly the same fashion. Thus, we have

$$\hat{p}_0=1, \quad \hat{p}_1=\cos\theta. \tag{9}$$

The starting point to examine the transport properties of the model will be the characteristic function (2), whose logarithm can be written as

$$\ln \hat{P}_\tau(\theta)=(T-q)\ln \hat{p}_N(\theta)+q\ln \hat{p}_{N+1}(\theta). \tag{10}$$

Expanding Eq. (10) in successive powers of θ, we obtain the cumulant expansion for the problem. Since this process tends to the sum of T independent identically distributed variables with finite variance, the central limit theorem applies and its distribution tends to a Gaussian. Thus, to analyze the asymptotic behavior of the process at large T and N, we only require the terms up to quadratic order in θ. Substituting the expansion of $\hat{p}_N(\theta)$ in Eq. (10), we find

$$\langle x\rangle=-\frac{\beta}{(1-\alpha)^2}[T(1-\alpha^N)+q\alpha^N(1-\alpha)] + (NT+q)\frac{\beta}{(1-\alpha)} \tag{11}$$

and

$$\langle x^2\rangle-\langle x\rangle^2=\frac{\beta^2}{(1-\alpha)^4}\{-T(1-\alpha^N)^2+q[(1-\alpha^N)^2 -(1-\alpha^{N+1})^2]\}-\frac{2}{(1-\alpha)^2}\left[\alpha-\frac{\beta^2}{(1-\alpha)^2}\right] \times\{T(1-\alpha^N)+q[(1-\alpha^{N+1})-(1-\alpha^N)]\} +(NT+q)\left(\frac{1+\alpha}{1-\alpha}\right)\left(1-\frac{\beta^2}{(1-\alpha)^2}\right), \tag{12}$$

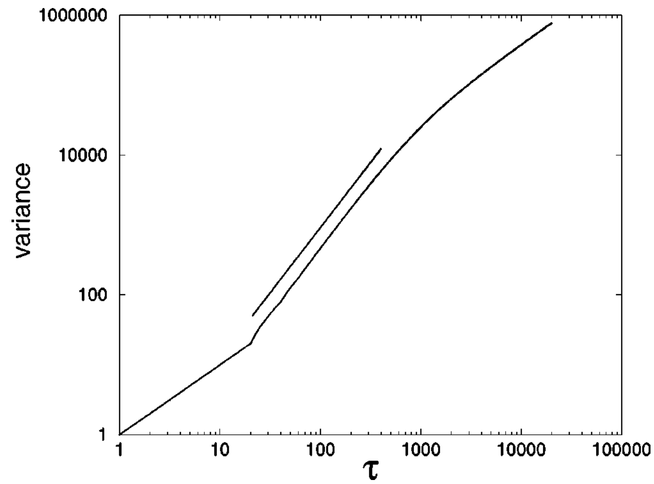

FIG. 2. σ^2 vs τ in a system with $T=20$ and $\varepsilon=\varepsilon'=0.95$. A nondiffusive regime is observed for times between T and $\tau_c\simeq 400$ followed by a slow return to diffusive behavior. The intermediate regime has a slope ~ 2.

where

$$\alpha=\tfrac{1}{2}(\varepsilon+\varepsilon'), \quad \beta=\tfrac{1}{2}(\varepsilon-\varepsilon'). \tag{13}$$

The evolution of the first moment for this particular process presents the following features. First of all, as expected, when $N=0$, $\langle x\rangle=0$ as a consequence of the symmetry of the initial conditions. This initial behavior persists for a time $\tau_c\sim T/(1-\alpha)$ as $\alpha\to 1$. After this time has elapsed, the drift in the system takes over, yielding

$$\langle x\rangle\simeq\frac{\beta}{(1-\alpha)}\left[\tau-\frac{T}{(1-\alpha)}\right]. \tag{14}$$

For the description of the behavior of the variance expressed in Eq. (12), we distinguish the following cases.

Case (i). $|\alpha|<1$: In this case for times $\tau\gg T$ the asymptotic behavior of $\sigma^2=\langle x^2\rangle-\langle x\rangle^2$ is diffusive in τ, plus a constant correction term induced by the persistence of the dispersion in the initial conditions

$$\sigma_\tau^2\simeq\frac{(1+\alpha)}{(1-\alpha)^3}[(1-\alpha)^2-\beta^2)]\tau+\left[\frac{\beta^2}{(1-\alpha)^4}-\frac{2\alpha}{(1-\alpha)^2}\right]T. \tag{15}$$

If either $\varepsilon=1$ or $\varepsilon'=1$, then $\beta^2=(1-\alpha)^2$ and the diffusive term vanishes identically, leaving only the constant term as the long-time limit for the variance. As mentioned above, in this situation either the left-moving or right-moving state is absorbing and the variance reflects the fluctuations of the process before entering the absorbing state. On the other hand, for $T<\tau<\tau_c$, the variance shows transient effects due to the persistence of the behavior of the initial conditions. The behavior during this transient period resembles that of the perfect-memory case $(\alpha=1)$, which is ballistic, as discussed below. Finally, given the initial conditions we are considering, when $\tau\sim T$, σ^2 is that of an ordinary random

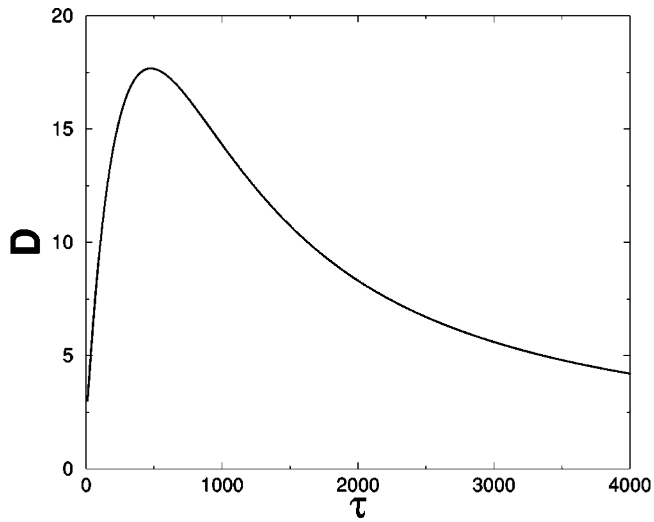

FIG. 3. Diffusion coefficient vs time for a system with $\varepsilon=0.95$, $\varepsilon'=1$, and $T=10$. A maximum resembling those observed in stochastic resonance is observed at $\tau\sim\tau_c\sim 400$.

walk over the initial period $(N=0)$. In Fig. 2, we show the various behaviors of the variance for the case in which $\alpha=0.95$, $\beta=0$, and $T=20$. This sequence of behaviors gives rise to a nonmonotonic behavior of the diffusion coefficient, $D=\sigma^2/\tau$ when $\beta\neq 0$. In Fig. 3, we show an example with $T=10$, $\varepsilon=0.95$, and $\varepsilon'=1$. There is a maximum value of the diffusion coefficient as a function of τ, which occurs at $\tau\sim\tau_c$. This maximum resembles the behavior of systems that exhibit the phenomenon of stochastic resonance [5], yet in this case it is due to the persistence of the fluctuations of initial conditions, combined with the boundedness of the variance at long times. Another type of nonmonotonic behavior of the diffusion coefficient occurs when α is near -1; this situation is discussed in case (iii).

Case (ii). $\alpha=1$: In this case, $\tau_c\to\infty$ and the process is nondiffusive for all $\tau>T$. This case corresponds to a drunkard's walk on a one-dimensional lattice with discrete time steps, which repeats exactly the same sequence of steps every T time units. Then, after the first period, the process becomes deterministic. This process actually occurs when using random number generators, where after a presumably large period T, the series repeats itself. To analyze this limit, we define $\lambda\equiv 1-\alpha$ and expand σ^2 in the neighborhood of $\lambda=0$ up to order λ^2. This procedure yields

$$\sigma^2=N^2T+2Nq+q\sim N^2T \quad \text{when} \quad T,N\gg 1. \tag{16}$$

In terms of τ, we may rewrite the expression above as

$$\sigma^2\sim\frac{\tau^2}{T}, \tag{17}$$

which shows that for finite T, the motion becomes essentially ballistic in the long-time limit. This result is easy to understand given the fact that the initial condition is a normal, unbiased, T-step random walk. After T steps, this initial random walk will have a mean-square displacement equal to T. The $\alpha=1$ process will then repeat exactly the same steps every period, giving rise to the same displacement at the end of each period; this results in the ballistic behavior predicted in Eq. (17).

Case (iii). $\alpha=-1$: This limit corresponds to a drunkard's walk on a one-dimensional lattice with discrete time steps that repeats exactly the same sequence of steps with opposite signs every T time units. This results in periodic motion with period $2T$. Thus, the variance remains bounded and it may be written as

$$\sigma^2=\tfrac{1}{2}[T-(-1)^N(T-2q)]. \tag{18}$$

For values of α near -1, this periodicity is again reflected in a transient regime for the diffusion coefficient in which it is nonmonotonic over the range of times $T<\tau<\tau_c$. D oscillates with maxima every two periods and relative amplitude decaying as time goes by. For $\tau>\tau_c$, D tends to a constant.

In summary, we present a simple generalization of the persistent random-walk model by considering arbitrarily delayed persistence among the states of a random walker, and assigning a step distribution to each state. We show how to calculate the probability distribution function for the position of this ''random walk with delayed state persistence'' by recasting it as the sum of T unit delay independent random processes. As an example, we study the transport properties of the random walk with delayed step persistence, for which we find that the distribution of positions tends asymptotically to a Gaussian distribution and obtain exact expressions for the transport coefficients of the model.

Aside from other applications of this model, tailored to specific physical systems, extensions to situations in which the state of the walker at step τ depends on the states of the walker at steps $\tau-T$ and $\tau-2T$ can be treated following essentially the same procedure. Interestingly, the procedure breaks down completely if we consider delays T and $2T+1$. This breakdown could be expected to manifest itself in the statistical properties of the process from the fact that in the large-T limit, there appears to be no way to express the position as a sum of T, short memory, independent random variables. On the other hand, on intuitive grounds, it is not clear how a process that depends on its states 28 763 and 57 526 steps ago will differ in its statistical features from one that depends on its states 28 763 and 57 527 steps ago. This question is presently under study.

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